Critical generation of soliton mode in a gauged O(3) sigma model with interpolating potential

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Abstract

We show that a new soliton mode is generated in a gauged sigma model with interpolating potential when the interpolation parameter decreases below a critical value. PACS:11.10.Kk; 11.10.Lm; 11.15.-q

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Recently a class of gauged O(3) sigma models in three dimensions have been studied [1–5] where a U(1) subgroup of the O(3) symmetry of the model was gauged by coupling the sigma model fields with a gauge field through the corresponding U(1) current. This is different from the minimal coupling via the topological current discussed previously [6]. The motivation behind the new gauged O(3) sigma models was to break the scale invariance of the self - dual solutions of the usual O(3) sigma model [7]. Initially the gauge field dynamics was assumed to be dictated by the Maxwell term [1]. Later the extension of the model with the Chern - Simons coupling was investigated [2]. A particular form of self - interaction was required to be included in these models in order to saturate the Bogomol'nyi bounds [8]. The form of the assumed self - interaction potential is of crucial importance. The minima of the potential determine the vaccum structure of the theory. The solutions change remarkably when the vaccum structure exhibits spontaneous breaking of the symmetry of the gauge group. Thus it was demonstrated that the observed degeneracy of the solutions of [1,2] is lifted when potentials with symmetry breaking minima were incorporated [4,5].

A generalisation of the models [1,2] was proposed in [3] where an adjustable real parameter v was introduced in the expression of the self - interaction potential. Detailed solutions of the model with the C - S coupling were provided. The parameter v interpolates between the symmetric and symmetry breaking phases. The inclusion of such adjustable potential in the corresponding Maxwell coupled model was conjectured, but not fully explored. In this letter we show that this generalisation indeed leads to certain important consequences. The most interesting outcome is a kind of 'critical phenomenon' where a new self - dual soliton mode emerges when the parameter v decreases below v = 1.

It will be useful to start with a brief review of the nonlinear O(3) sigma model [7]. The lagrangian of the model is given by,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi} \tag{1}$$

Here $\boldsymbol{\phi}$ is a triplet of scalar fields constituting a vector in the internal space with unit norm

$$\phi_a = \mathbf{n_a} \cdot \boldsymbol{\phi}, \ (a = 1, 2, 3) \tag{2}$$

$$\boldsymbol{\phi} \cdot \boldsymbol{\phi} = \phi_a \phi_a = 1 \tag{3}$$

The vectors $\mathbf{n}_a$ constitute a basis of unit orthogonal vectors in the internal space. We work in the Minkowskian space - time with the metric tensor diagonal, $g_{\mu\nu} = (1, -1, -1)$.

The finite energy solutions of the model (1) satisfies the boundary condition

$$\lim \phi^a \;=\; \phi^a_{(0)} \tag{4}$$

at physiacal infinity. The condition (4) corresponds to one point compactification of the physical infinity. The physical space R_2 becomes topologically equivalent to S_2 due to this compactification. The static finite energy solutions of the model are then maps from this sphere to the internal sphere. Such solutions are classified by the homotopy [9]

$$\Pi_2(S_2) = Z \tag{5}$$

We can construct a current

$$K_\mu = \frac{1}{8\pi} \epsilon_{\mu\nu\lambda} \boldsymbol{\phi} \cdot (\partial^\nu \boldsymbol{\phi} \times \partial^\lambda \boldsymbol{\phi}) \tag{6}$$

which is conserved irrespective of the equation of motion. The corresponding charge

$$\begin{aligned} T &= \int d^2\mathbf{x} K_0 \\ &= \frac{1}{8\pi} \int d^2\mathbf{x} \epsilon_{ij} \boldsymbol{\phi} \cdot (\partial^i \boldsymbol{\phi} \times \partial^j \boldsymbol{\phi}) \end{aligned} \tag{7}$$

gives the winding number of the mapping (5) [10].

In the class of gauged models of our interest here a U(1) subgroup of the rotation symmetry of the model (1) is gauged. We chose this to be the SO(2) [U(1)] subgroup of rotations about the 3 - axis in the internal space. The Lagrangian of our model is given by

$$\mathcal{L} = \frac{1}{2} D_\mu \boldsymbol{\phi} \cdot D^\mu \boldsymbol{\phi} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + U(\boldsymbol{\phi}) \tag{8}$$

$D_\mu \boldsymbol{\phi}$ is the covariant derivative given by

$$D_\mu \boldsymbol{\phi} = \partial_\mu \boldsymbol{\phi} + A_\mu \mathbf{n}_3 \times \boldsymbol{\phi} \tag{9}$$

The SO(2) (U(1)) subgroup is gauged by the vector potential A_μ whose dynamics is dictated by the Maxwell term. Here $F_{\mu\nu}$ are the electromagnetic field tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{10}$$

$U(\boldsymbol{\phi})$ is the self - interaction potential required for saturating the self - dual limits. We chose

$$U(\boldsymbol{\phi}) = -\frac{1}{2}(v - \phi_3)^2 \tag{11}$$

where v is a real parameter. Substituting v = 0 we get back the model of [5] whereas v = 1 gives the model of [1].

We observe that the minima of the potential arise when ,

$$\phi_3 = v \tag{12}$$

which is equivalent to the condition

$$\phi_1^2 + \phi_2^2 = 1 - v^2 \tag{13}$$

on account of the constraint (3). The values of v must be restricted to

$$|v| \leq 1 \tag{14}$$

The condition (13) denotes a latitudinal circle (i.e. circle with fixed latitude) on the unit sphere in the internal space. By varying v from -1 to +1 we span the sphere from the south pole to the north pole. It is clear that the finite energy solutions of the model must satisfy (13) at physical infinity. For $v \neq 1$ this boundary condition corresponds to the spontaneous breaking of the symmetry of the gauge group and in the limit

$$|v| \rightarrow 1 \tag{15}$$

the asymmetric phase changes to the symmetric phase. We call the potential (11) interpolating in this sense. In the asymmetric phase the soliton solutions are classified according to the homotopy

$$\Pi_1(S_1) = Z \tag{16}$$

instead of (5). In the symmetric phase, however, this new topology disappears and the solitons are classified according to (5) as in the usual sigma model (1). In the following we will observe a remarkable fallout of this change of topology. The fundamental solitonic mode n = 1 (n being the vorticity) ceases to exist in the symmetric phase. The modes corresponding to n = 2 onwards still persist but the magnetic flux associated with them ceases to remain quantised.

In the asymmetric phase the vorticity is the winding number i.e. the number of times by which the infinite physical circle winds over the latitudinal circle (13). Associated with this is a uniqe mapping of the internal sphere where the degree of mapping is usually fractional. By inspection [1] we construct a current

$$K_\mu = \frac{1}{8\pi}\epsilon_{\mu\nu\lambda}[\boldsymbol{\phi}\cdot D^\nu\boldsymbol{\phi}\times D^\lambda\boldsymbol{\phi} - F^{\nu\lambda}(v-\phi_3)] \tag{17}$$

generalising the topological current (6). The current (17) is manifestly gauge invariant and differs from (6) by the curl of a vector field [1]. The conservation principle

$$\partial_\mu K^\mu = 0 \tag{18}$$

thus automatically follows from the conservation of (6). The corresponding conserved charge is

$$T = \int d^2x K_0 \tag{19}$$

Using (17) and (19) we can write

$$T = \int d^2x[\frac{1}{8\pi}\epsilon_{ij}\boldsymbol{\phi}\cdot(\partial^i\boldsymbol{\phi}\times\partial^j\boldsymbol{\phi})] - \frac{1}{4\pi}\int_{boundary}(v-\phi_3)A_\theta r d\theta \tag{20}$$

where r,θ are polar coordinates in the physical space and $A_\theta = \mathbf{e}_\theta\cdot\mathbf{A}$. Using the boundary condition (12) we find that T is equal to the degree of the mapping of the internal sphere. Note that this situation is different from [3] where the topological charge usually differs from the degree of the mapping. In this context it is interesting to observe that the current (17) is not unique because we can always add an arbitrary multiple of

$$\frac{1}{8\pi}\epsilon_{\mu\nu\lambda}F^{\nu\lambda} \tag{21}$$

with it without affecting its conservation. We chose (17) because it generates proper topological charge.

The Euler - Lagrange equations of the system (8) is derived subject to the constraint (3) by the Lagrange multiplier technique

$$D_\nu(D^\nu\boldsymbol{\phi}) = [D_\nu(D^\nu\boldsymbol{\phi})\cdot\boldsymbol{\phi}]\boldsymbol{\phi} + \mathbf{n}_3(v-\phi_3) + (v-\phi_3)\phi_3\boldsymbol{\phi} \tag{22}$$

$$\partial_\nu F^{\nu\mu} = j^\mu \tag{23}$$

where

$$j^\mu = -\mathbf{n}_3\cdot\mathbf{J}^\mu \ \ and\ \ \mathbf{J}^\mu = \boldsymbol{\phi}\times D^\mu\boldsymbol{\phi} \tag{24}$$

Using (22) we get

$$D_\mu\mathbf{J}^\mu = -(v-\phi_3)(\mathbf{n}_3\times\boldsymbol{\phi})\phi_3 \tag{25}$$

¿From (23) we find,for static configurations

$$\nabla^2 A^0 = -A^0(1-\phi_3^2) \tag{26}$$

¿From the last equation it is evident that we can chose

$$A^0 = 0 \tag{27}$$

As a consequence we find that the excitations of the model are electrically neutral.

We now construct the energy functional from the symmetric energy - momentum tensor. The energy

$$E = \frac{1}{2}\int d^2\mathbf{x}\left[D_0\boldsymbol{\phi}\cdot D_0\boldsymbol{\phi} - D_i\boldsymbol{\phi}\cdot D^i\boldsymbol{\phi} + (v-\phi_3)^2 - 2(F_0^\sigma F_{0\sigma} - \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma})\right] \tag{28}$$

. For static configuration and the choice $A^0 = 0$, E becomes

$$E = \frac{1}{2}\int d^2x[(D_i\boldsymbol{\phi})\cdot(D_i\boldsymbol{\phi}) + F_{12}^2 + (v-\phi_3)^2] \tag{29}$$

Several observations about the finite energy solutions can be made at this stage from (29). By defining

$$\psi = \phi_1 + i\phi_2 \tag{30}$$

we get

$$D_i\boldsymbol{\phi}\cdot D_i\boldsymbol{\phi} = |(\partial_i + iA_i)\psi|^2 + (\partial_i\phi_3)^2 \tag{31}$$

The boundary condition (13) dictates that

$$\psi \approx (1-v^2)^{\frac{1}{2}}e^{in\theta} \tag{32}$$

at infinity. ¿From (29) we observe that for finite energy configurations we require

$$\mathbf{A} = \mathbf{e}_\theta\frac{n}{r} \tag{33}$$

on the boundary. This scenario is exactly identical with the observations of [5] and leads to the quantisation of the magnetic flux

$$\Phi = \int Bd^2x = \int_{boundary} A_\theta r d\theta = 2\pi n \tag{34}$$

The basic mechanism leading to this quantisation remains operative so far as v is less than 1. At v = 1, however, the gauge field $\mathbf{A}$ becomes arbitrary on the boundary except for the requirement that the magnetic field B should vanish on the boundary. We will presently show that not all the vortices present in the broaken phase survives this demand. Specifically, the n = 1 vortex becomes inadmissible.

Now the search for the self - dual conditions proceed in the usual way. We rearrange the energy functional as

$$E = \frac{1}{2}\int d^2x[\frac{1}{2}(D_i\boldsymbol{\phi} \pm \epsilon_{ij}\boldsymbol{\phi}\times D_j\boldsymbol{\phi})^2 + (F^{12} \mp (v-\phi_3))^2] \pm 4\pi T \tag{35}$$

Equation (35) gives the Bogomol'nyi conditions

$$D_i\boldsymbol{\phi} \pm \epsilon_{ij}\phi \times D_j\boldsymbol{\phi} = 0 \tag{36}$$

$$F_{12} \mp (v - \phi_3) = 0 \tag{37}$$

which minimize the energy functional in a particular topological sector, the upper sign corresponds to +ve and the lower sign corresponds to -ve value of the topological charge.

We will now turn towards the analysis of the self - dual equations using the rotationally symmetric ansatz [11]

$$\begin{aligned}\phi_1(r,\theta) &= \sin g(r)\cos n\theta \\ \phi_2(r,\theta) &= \sin g(r)\sin n\theta \\ \phi_3(r,\theta) &= \cos g(r) \\ \mathbf{A}(r,\theta) &= -\mathbf{e}_\theta \frac{na(r)}{r}\end{aligned} \tag{38}$$

From (12) we observe that we require the boundary condition

$$g(r) \to \cos^{-1} v \ \text{ as } \ r \to \infty \tag{39}$$

and equation (33) dictates that

$$a(r) \to -1 \ \text{ as } \ r \to \infty \tag{40}$$

Remember that equation (33) was obtained so as the solutions have finite energy. Again for the fields to be well defined at the origin we require

$$g(r) \to 0 \ \text{ or } \ \pi \ \text{ and } \ a(r) \to 0 \ \text{ as } \ r \to 0 \tag{41}$$

Substituting the Ansatz(38) into (36) and (37) we find that

$$g'(r) = \pm\frac{n(a+1)}{r}\sin g, \tag{42}$$

$$a'(r) = \pm\frac{r}{n}(v - \cos g) \tag{43}$$

where the upper sign holds for +ve T and the lower sign corresponds to -ve T.Equations (42) and (43) are not exactly integrable. However, the following analysis of the boundary value problem defined by (42) and (43) with (39) to (41) in the line of [1] reveals general nature of the admissible solutions.

Using the Ansatz (38) we can explicitly compute the topological charge T by performing the integration in (19).The result is

$$T = -\frac{n}{2}[\cos g(\infty) - \cos g(0)] - \frac{1}{2}[v - \cos g(\infty)] \tag{44}$$

The second term of (44) vanishes due to the boundary condition (39). Also, when g(0) = 0,

$$T = \frac{n}{2}(1 - v) \tag{45}$$

and, when g(0) = π,

$$T = -\frac{n}{2}(1 + v) \tag{46}$$

It is evident that T is in general fractional. Due to (20) it is equal to the degree of mapping of the internal sphere. This can also be checked explicitly.

¿From the above analysis we find that g(0) = 0 corresponds to +ve T and g(0) = π corresponds to -ve T. We shall restrict our attention on negetive T mainly for comparision with [1]. The boundary value problem of interest is then

$$g'(r) = -\frac{n(a+1)}{r}\sin g \tag{47}$$

$$a'(r) = -\frac{r}{n}(v - \cos g) \tag{48}$$

with

$$\begin{gathered} g(0) = \pi, a(0) = 0 \\ f(\infty) = \cos^{-1} v, a(\infty) = 0 \end{gathered} \tag{49}$$

In addition we require $a'(r) \to 0$ as $r \to \infty$. Looking at (47), (48) and (49), the last condition appears to be a consistency condition.

The behaviour of the functions near r = 0 can be easily derived as in [1]

$$g(r) \approx \pi + Ar^n \tag{50}$$

$$a(r) \approx -\frac{r^2}{2n}(1+v) \tag{51}$$

Here A is an arbitrary constant which fixes the values of g and a at infinity. There is a critical value of A, $A = A^{crit}$ for which the boundary conditions are satisfied [5]. If the value of A is larger than A^{crit} the conditions at infinity are overshooted, whereas, if the value is smaller than the critical value g(r) vanishes asymptotically after reaching a maximum. The situation is comparable with similar findings elsewhere [12].

We are specifically concerned about the limit $v \longrightarrow 1$ because of our interest in the transition from asymmetric to symmetric vaccum. First observe that according to (42) and (43) $\cos g$ cannot exceed v in the finite region because in that event a(r) becomes increasing function and g(r) overshoots the boundary value. $a'(r)$ is then always -ve, so that a(r) is a decreasing function. Let $a(\infty) = \alpha$. Repeating the analysis of [1] we can prove that

$$-1 \leq \alpha \leq 0 \tag{52}$$

Also, in the limit $v \longrightarrow 1$ g is small and the equation (47) simplifies to

$$g'(r) = -\frac{n(a+1)}{r}g \tag{53}$$

which gives on integration,

$$g(r) \approx Cr^{-n(\alpha+1)} \tag{54}$$

with C being a positive constant. Note that α = -1 is admissible in the limit $v \longrightarrow 1$ as long as (v - 1) is nonzero. This is because $g(\infty)$ is then small but nonzero and the boundary condition on g can be adjusted by C. However, if we put v = 1 then we require g to vanish on the boundary. This requirement dictates that

$$\alpha > -1 \tag{55}$$

on the boundary. Now expanding the equation (43) we get

$$a'(r) = -\frac{r}{2n}(g^2 - g(\infty)^2) \tag{56}$$

So far as $g(\infty)$ is small but different from zero, the limit

$$g \to g(\infty) \tag{57}$$

is sufficient to ensure the condition on $a'(r)$. But when $g(\infty) = 0$ we have

$$a'(r) = -\frac{C^2}{2n} r^{1-2n(\alpha+1)} \tag{58}$$

Thus a can only converge if

$$2n(\alpha + 1) - 1 > 1 \tag{59}$$

i.e.

$$\alpha > \frac{1}{n} - 1 \tag{60}$$

Clearly n = 1 is not permissible. The mode n = 1 is quenched when v becomes equal to 1. Again, in this limit α is arbitrary. So the magnetic flux associated with the surviving modes is no longer quantised. Thus we arrive at our principal result that in the model (8) with the interpolating potential (11), the soliton mode with vorticity 1 is generated when the parameter v decreases below the critical value v = 1. Along with this, the magnetic flux associated with the modes present in the v = 1 phase become quantised.

We have considered a gauged O(3) sigma model with the gauge field dynamics determined by the Maxwell term. An interpolating potential was included as in [3]. This potential depends on a free parameter, the variation of which effects transition from the symmetric to asymmetric phase. We have discussed the transition of the associated topology of the soliton solutions and demonstrated that a new soliton mode emerges when the parameter decreases below a critical value. This conclusion follows from an analysis of the self - dual equations at the 'critical point' (i.e. at the transition point). The self - dual equations are, however, not exactly solvable. But they can be studied numerically to trace out the growth of the generated mode in detail. Such analysis may also be interesting from the point of view of application, particularly in condensed matter physics.

REFERENCES

[1] B.J. Schroers, Phys. Lett. B 356, 291 (1995).

[2] P.K. Ghosh and S.K. Ghosh, Phys. Lett. B366, 199 (1996).

[3] K. Kimm, K. Lee and T. Lee, Phys. Rev. D 53 (1996) 4436.

[4] P. Mukherjee, Phys. Lett. B 403, 70 (1997).

[5] P. Mukherjee, Phys. Rev. D 58 (1998) 105025.

[6] F. Wilczek and A. Zee, Phys. Rev. Lett. 51 (1983) 2250; M. Bowick, D. Karabali and L. C. R. Wijewardhana, Nucl. Phys. B 271 (1986) 417; R. Banerjee, Nucl. Phys. B 419 (1994) 611.

[7] A.A. Belavin and A.M. Polyakov, JETP Lett. 22, 245 (1975).

[8] E.B. Bogomol'nyi, Sov. J. Nucl. Phys. 24, 449 (1976).

[9] F. J. Hilton, An introduction to Homotopy theory (Cambridge University Press, Cambridge, England, 1953.

[10] R.Rajaraman, Solitons and Instantons (North Holland Publishing Company, Amsterdam, 1989).

[11] Y.S. Wu and A. Zee, Phys. Lett. B147, 325 (1984).

[12] R. Jackiw, K. Lee and E. Weinberg, Phys. Rev. D 42, 3488 (1990).